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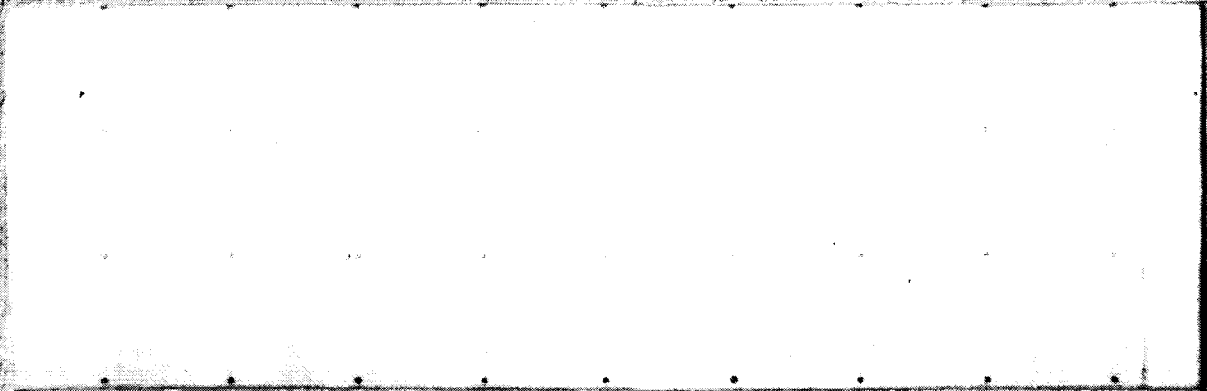
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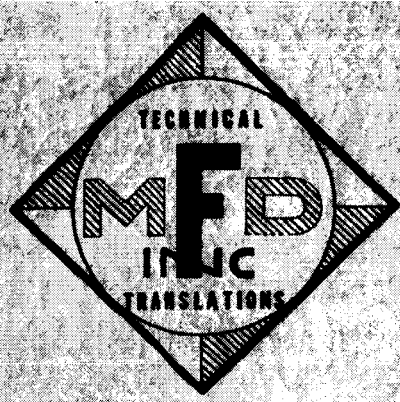
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Morris D. Friedman

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2 Pine Street

West Concord, Mass.

Neutralization Method in Solving Wave Problems

N. N. ANDREEV

Commemorative volume honoring the 70th Birthday of
A. I. Ioffe, Izd. Akad. Sci. USSR
Moscow, 1950

The method, explained below, of neutralizing the original differential equation can often be applied in the approximate solution of different wave problems.

The idea of applying neutralization directly to the differential equation is not new. Duhamel [1] showed that the equation of the propagation of sound in a cylindrical pipe with section of any form can be reduced to the form:

$$\frac{\partial^2 \bar{\varphi}}{\partial t^2} = c^2 \frac{\partial^2 \bar{\varphi}}{\partial x^2}$$

where

$$\bar{\varphi} = \frac{1}{S} \int_S \varphi dS$$

Obviously, this equation gives an approximate representation of sound propagation in such a pipe. Liouville [2] used neutralization of the wave equation over a sphere to derive the well known Poisson solution [3].

Such neutralization always changes the number of independent variables. In the classical Poisson case, it would permit finding a rigorous solution of the complete

problem; however, the boundary conditions do not permit this to be done. Frequently, neutralization is useful as a method of approximate solution; for example, for a pipe of variable section (loudspeaker) it leads to the well-known Webster equation which is now standard in all books on acoustics but usually derived by other methods [4]. However, the solution of the neutralized equation is sufficiently approximate only when the largest dimension of the section is less than a half-wave. How to obtain further, better approximations is shown by two examples in the present work.

Propagation of sound waves near a sloping beach

The problem which we consider, first of all, can be formulated as follows (Fig. 1). In a wedge-shaped volume of water, sound waves of frequency ω arrive and are reflected from the plane bottom OA, which we will consider to be completely solid (rocky bottom), and fields of standing waves are generated from the free surface OB. The problem consists of finding the sound pressure in this field.

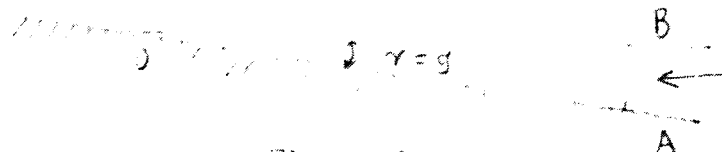


Figure 1.

Let us select cylindrical coordinates and let us

assume that the phenomenon depends only on two coordinates.

The propagation equation is in this case:

$$(1) \quad k^2 \varphi + \frac{1}{r} \left[\frac{\partial}{\partial r} r \left(\frac{\partial \varphi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \varphi}{\partial \gamma^2} \right]$$

in which the angle γ is measured from the surface of the water and takes the value π on the bottom.

The pressure and the velocity of the motion of the water particles are determined from the solution of (1) by means of the formulas:

$$(2) \quad p = \gamma \omega \rho_0 \varphi; \quad v_r = - \frac{\partial \varphi}{\partial r}$$

$$v_\gamma = - \frac{1}{r} \frac{\partial \varphi}{\partial \gamma}$$

Hence it is seen that the boundary conditions of our problem can be formulated thus:

$$(3) \quad \begin{aligned} \varphi &= 0 & \text{for } \gamma &= 0 \\ \frac{\partial \varphi}{\partial \gamma} &= 0 & \text{for } \gamma &= \pi \end{aligned}$$

Let us introduce the new quantity $\bar{\varphi}$ in (1) which is related to φ through:

$$(4) \quad \bar{\varphi} = \int_0^\pi \varphi d\gamma$$

and, for brevity, we will call the operation of obtaining $\bar{\varphi}$ from φ neutralization. We can introduce another definition:

$$\bar{\varphi} = \int_0^\pi \varphi r d\gamma$$

and then $\bar{p}$ and $\bar{v}_r$ would be the total force and volume velocity acting on a segment of unit width in the section $r = \text{const.}$ However, it appears to be inconvenient for computations. Both these definitions do not correspond to the data above: generally, different definitions of neutralization should be used in different problems, the most adequate to the problem being chosen. Application of (4) transforms (1) and (2) into:

$$(5) \quad k^2 \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\varphi}}{\partial r} + \frac{1}{r^2} \left| \frac{\partial \varphi}{\partial \gamma} \right|^2 = 0$$

$$\bar{p} = \gamma \omega \rho_0 \bar{\varphi} ; \quad \bar{v}_r = - \frac{\partial \bar{\varphi}}{\partial r}$$

Let us assume that we consider the region of the field, $r < R$, in which the vertical section is small compared to the wave length. Then expanding the desired solution in terms of γ , we can be limited to a small number of terms. Let us put

$$(6) \quad \varphi = \gamma R_1 - \gamma^3 R_3$$

where R_1, R_3 are functions of r .

Assumption (6) permits reduction of (5) to a form containing only the one unknown, $\bar{\varphi}$. We have

$$\bar{\varphi} = \frac{1}{3} \gamma^2 R_1 - \frac{1}{4} \gamma^4 R_3$$

$$\left(\frac{\partial \varphi}{\partial \gamma} \right)_{\gamma=0} = R_1 + 3 \gamma^2 R_3 = 0$$

Hence, expressing R_1 and R_3

$$R_1 = \frac{12\bar{\varphi}}{5\epsilon^2} \quad ; \quad R_3 = -\frac{4\bar{\varphi}}{5\epsilon^4}$$

$$\varphi = \gamma \bar{\varphi} \frac{12}{5\epsilon^2} \left(1 - \frac{\gamma^2}{3\epsilon^2} \right)$$

$$\left(\frac{\partial \varphi}{\partial r} \right)_{r=0} = \frac{12\bar{\varphi}}{5\epsilon^2}$$

and the original equation becomes:

$$\left(k^2 - \frac{12}{5\epsilon^2 r^2} \right) \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\varphi}}{\partial r} = 0$$

or

$$(7) \quad \left(1 - \frac{12}{5\epsilon^2 (kr)^2} \right) \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial (kr)^2} + \frac{1}{kr} \frac{\partial \bar{\varphi}}{\partial (kr)} = 0$$

i.e., we obtain Bessel's equations. Denoting

$$m^2 = \frac{12}{5\epsilon^2}$$

i.e., putting

$$m \sim \frac{1.55}{\epsilon}$$

we can write the solution as

$$(8) \quad \bar{\varphi} = A I_m(kr) + B Y_m(kr)$$

The requirement that the pressure be finite at $r = 0$ yields $B = 0$; the desired solution for φ is then

$$(9) \quad \varphi = A \left(\gamma - \frac{\gamma^3}{3\epsilon^2} \right) I_m(kr)$$

As is known, the greater the m , the greater the part of I_m close to zero and, therefore, the greater the region near the beach has a pressure field near zero.

But the magnitude of m is inversely proportional to the angle α . Our solution has a simple and graphical form. For example, for $\alpha = 0.155$ or, about 9° , zero values of pressure for (kr) of order 6.

It would be of interest to confirm these formulas on a model where there is the possibility of realizing a solid bottom (fig. 2).

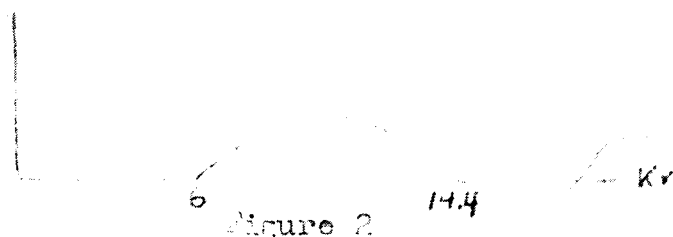


Figure 2

In actual cases, the bottom cannot be considered solid. This circumstance can be taken into account by changing the boundary condition for $\gamma = g$. For this we put¹ for $\gamma = g$:

$$(10) \quad Z = \left(\frac{p}{v_n} \right)_{\gamma=g}$$

where Z is the so-called impedance (wave resistance) of the bottom. This condition has to be written in the form:

$$\left(\alpha \frac{\partial \varphi}{\partial \gamma} + \varphi \right)_{\gamma=g} = 0 ; \quad \alpha = - \frac{rZ}{j\omega \rho_0}$$

As a consequence of these conditions the coefficient of $1/(kr)^2$ becomes a complex function of r and the solution is difficult and, in all cases, loses graphicalness.

The case of propagation in an even layer of water with a solid bottom is solved very simply. Again we chose the

1. This assumption has limited application. On the bounds of applicability see [5].

neutralization definition in the previous form (fig. 3):



$$\bar{\varphi} = \int_0^h \varphi dx$$

Figure 3
and we arrive at the neutralized equation

$$0 = k^2 \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial x^2} - \left(\frac{\partial \bar{\varphi}}{\partial x} \right)_{z=h}$$

which with the assumption

$$\varphi = R_1 z + R_3 z^3$$

reduces to

$$\bar{\varphi} \left(k^2 - \frac{12}{5h^2} \right) + \frac{\partial^2 \bar{\varphi}}{\partial x^2} = 0$$

and the solution for φ is:

$$(11) \quad \varphi = z \left(1 - \frac{z^2}{3h^2} \right) \exp \left[-j \sqrt{k^2 - \frac{12}{5h^2}} \right]$$

In the considered case, the exact solution is obtained easily:

$$\varphi = \exp \left[-j \sqrt{k^2 - \left(\frac{\pi}{2h} \right)^2} \right] \sin \frac{z}{2h}$$

Our approximate solution coincides well with the exact because π is close to $12/5$.

The characteristic singularity of the solution is the presence of the exponential decrease in pressure ($j\omega\rho_0\varphi$) as far as propagation of the waves if the condition is maintained:

$$k^2 < \frac{12}{5h^2}$$

Let us consider one more example of neutralization

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which leads to the Webster equation, convenient in practice, wherein the nature of the concept of forming the approximation appears. Let us consider the circular tube of variable section $S(x)$ (body of revolution around the x axis) and we choose the neutralization along the section in the form

$$\bar{\varphi} = \frac{1}{S} \int_S \varphi \cdot dS$$

which Liouville used.

Starting from the equation

$$k^2 \varphi + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

we apply the above-mentioned operation and find:

$$0 = k^2 \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial x^2} + \frac{1}{S} \int_S \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} \right) dS$$

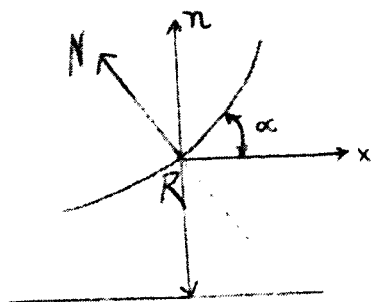
The last integral can be transformed into a line integral in which we find

$$\int_S \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z} \right) dS = \int_L \frac{\partial \varphi}{\partial n} dL$$

where n is the external normal to L lying in a plane perpendicular to the x axis. Taking into account that our pipe is a body of revolution and, moreover, assuming that the sound field maintains axial symmetry, we see that $\partial \varphi / \partial n$ is independent of location on the contour L and equals zero on the axis. This leads us to the equation:

$$(12) \quad 0 = k^2 \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial x^2} - \frac{2}{R} \left(\frac{\partial \varphi}{\partial n} \right)_L$$

The boundary condition on L for given $x = R$ is (fig. 4):



$$\left(\frac{\partial \varphi}{\partial n} \right)_L = 0$$

Expressing this condition through $\frac{\partial \varphi}{\partial n}$, $\frac{\partial \varphi}{\partial x}$ and observing that

$$\tan \alpha = \frac{\partial R}{\partial x} = \frac{1}{L} \frac{\partial S}{\partial x}$$

Figure 4

we find from the expression

$$\left(\frac{\partial \varphi}{\partial n} \right)_L = \left(\frac{\partial \varphi}{\partial x} \right)_L \sin \alpha - \left(\frac{\partial \varphi}{\partial n} \right)_L \cos \alpha = 0$$

that is

$$\left(\frac{\partial \varphi}{\partial n} \right)_L = \left(\frac{\partial \varphi}{\partial x} \right)_L \tan \alpha = \frac{1}{L} \left(\frac{\partial \varphi}{\partial x} \right)_L \frac{\partial S}{\partial x}$$

for $r = R$.

The basic equation now becomes

$$(13) \quad 0 = k^2 \bar{\varphi} + \frac{\partial^2 \bar{\varphi}}{\partial x^2} + \frac{1}{S} \frac{\partial S}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right)_L$$

and this equation differs from the Webster equation (for a body of revolution) only in the form of the last term.

Let us recall that the Webster equation is

$$0 = k^2 \varphi + \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{S} \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x}$$

Therefore, the one difference is that the last term

in the exact equation (13) contains the value of the velocity $\partial\phi/\partial x$ on the boundary, while this value in the Webster equation is the neutralized $\overline{\frac{\partial\phi}{\partial x}} = \frac{\partial\bar{\phi}}{\partial x}$.

Obviously, it is necessary to select sufficiently long wave lengths for this. Usually, there are other conditions: absence of large inclinations between the loudspeaker walls and the axis; however, this difficulty is from the derivation. There are assumed, of course, smoothness of pressure along a section, and for sharp boundaries and sufficiently long waves.

It is necessary to expect the neutralization method, under the condition of coincidence of boundaries with the coordinates of the surface, to permit finding approximate equations in such cases as, for example, plane and curved plates.

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